Group Report

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On the Accuracy and Resolution of Radar Signals

25 June 196[illegible]

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ON THE ACCURACY AND RESOLUTION OF RADAR SIGNALS

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Group 28

GROUP REPORT 1964-23

25 JUNE 1964

LEXINGTON MASSACHUSETTS

ABSTRACT

Information matrices are derived for estimates of the range parameters of moving targets as obtained by combining *apriori* information (if available) with reflected radar signals observed in the presence of additive white Gaussian noise. The inverse of the information matrix provides a lower bound on the covariance matrix of any unbiased parameter estimates. This bound can be approached with a high signal-to-noise ratio and optimum data processing (matched filters).

Arbitrary frequency modulation, amplitude modulation and target motion as well as various assumptions on processing the R-F phase are considered. The multiple target case enables investigation of a signal's resolution ability as well as its accuracy potentials. Results for a carrier frequency much greater than the effective signal bandwidth are obtained as a special case.

A main purpose of the report is the reduction of the original radar problem to a linear model which is equivalent in the sense of having the same information matrix. These models provide valuable insight into the relative effects of multiple targets, choice of modulation, *apriori* information, and assumptions regarding R-F phase and bandwidth. The linear equivalent model also leads to a valuable computational algorithm for investigations using digital or hybrid computers. The various special cases of interest are obtained by simple modifications of the general case and thus the algorithm can provide a very versatile tool for evaluating and designing radar signals.

Accepted for the Air Force
Franklin C. Hudson, Deputy Chief
Air Force Lincoln Laboratory Office

1. INTRODUCTION

We derive general formulae for the accuracies with which the range parameters of moving targets can be estimated from reflected radar signals observed in the presence of white noise. The single target case has already been investigated by many authors. (See for example, Refs. 1 and 2. Reference 1 contains an extensive bibliography.) In this sense, our work on the single target case is a rehash of the previous studies in a more general form. However, our motivation is not to write general formulae, but rather to convey a simplified picture of a complex problem. We pursue this goal by developing linear parameter estimation models which are equivalent to the original radar problem. These simple models delineate the relative effects of amplitude and frequency modulations, assumptions on how the R-F phase is processed, and *apriori* information. They provide a convenient starting point for the design of optimum modulations. Of equal if not greater importance, they suggest a simple computational procedure for evaluating performance and hence for conducting an experimental signal design.

The formulae for the multiple target case enable investigation of the resolution ability of a radar signal by determining how well the parameters of several targets can be simultaneously estimated from observing the sum of the radar signals reflected from the various targets. As in the single target case, the goal is an equivalent linear model which leads to both understanding and a convenient computational algorithm.

Our analysis is a straightforward application of certain results of classical statistics; the Cramer-Rao inequality and linear regression theory. We treat unknown parameters corresponding to signai amplitude and phase the same way as the unknown parameters of the target's motion.

In Sec. 2 we review the necessary statistical theory and in Sec. 3, apply it to a general radar model. In Sec. 4, the effects of unknown phase and amplitude parameters are discussed in terms of a matrix partitioning which removes these nuisance parameters from the problem. Equivalent linear models for various single target cases are presented in Sec. 5 while the multiple target case is discussed in Sec. 6. A general discussion is given in Sec. 7.

2. ERROR ANALYSIS FOR NONLINEAR PARAMETER ESTIMATION

Consider the following problem. We observe

$$z(t) = y(t, \underline{\xi}^{o}) + \eta(t) \qquad 0 \leq t \leq T \tag{2.1}$$

where $y(t, \underline{\xi})$ is a known function of time t and the parameter column vector $\underline{\xi}$. $\underline{\xi}^{o}$ is a fixed but unknown value of $\underline{\xi}$. $\eta(t)$ is a zero mean, stationary, white Gaussian stochastic process with two-sided noise power, σ^2. Unbiased estimates of $\underline{\xi}^{o}$ are to be made using the observations and whatever *a priori* information is available. We are interested in the covariance matrix of the errors in the estimates.

The following notational convention is used repeatedly. If $x(t, \underline{\psi})$ is some function of time t and an unknown parameter vector $\underline{\psi}$, then

$$\mathfrak{F}(x/\underline{\psi}) = \left.\begin{bmatrix} \dfrac{\partial x(t, \underline{\psi})}{\partial \psi_1} \\[2ex] \dfrac{\partial x(t, \underline{\psi})}{\partial \psi_2} \\ \vdots \end{bmatrix}\right|_{\underline{\psi} = \underline{\psi}^{o}} . \tag{2.2}$$

Thus $\mathfrak{F}(x, \underline{\psi})$ is the column vector formed from the partial derivatives of $x(t, \underline{\psi})$ with respect to $\psi_1, \psi_2, \cdots$, the elements of $\underline{\psi}$. The partial derivatives are evaluated at some chosen value of $\underline{\psi}$ denoted by $\underline{\psi}^{o}$.

Consider first a discrete time version of Eq. (2.1) wherein we make N observations

$$z(n) = y(n, \underline{\xi}^{o}) + \eta(n) \qquad n = 1, \cdots, N$$

where $\eta(n)$ is some discrete time stochastic process. Let

$$F(z(1), \cdots, z(N), \underline{\xi}^{o}) = F(\underline{z}, \underline{\xi}^{o})$$

be the cumulative distribution function of the observations. Let $\hat{\underline{\xi}}$ be any unbiased estimate of $\underline{\xi}^{o}$, made from the $z(n)$, $n = 1, \cdots, N$. Define $\hat{\Sigma}$ as the covariance matrix of $\underline{\xi}^{o} - \hat{\underline{\xi}}$; that is

$$\hat{\Sigma} = E[(\underline{\xi}^{o} - \hat{\underline{\xi}})(\underline{\xi}^{o} - \hat{\underline{\xi}})']$$

where the prime denotes transpose. If we define

$$B = \int \mathfrak{F}(\log F/\underline{\xi})\, \mathfrak{F}'(\log F/\underline{\xi})\, dF(\underline{z}, \underline{\xi}) \tag{2.3}$$

then under sufficient regularity conditions, the Cramer-Rao or Information inequality states

$$\hat{\Sigma} \geq B^{-1} \tag{2.4}$$

where the matrix inequality implies $\hat{\Sigma} - B^{-1}$ is a positive definite matrix. References 3, 4 and 5 are three among many which derive this result. (In Refs. 4 and 5 see discussions on efficiency of estimation.) When $\eta(n)$ is a zero mean, discrete, stationary white Gaussian process with variance σ^2, Eq. (2.3) becomes

$$B = \frac{1}{\sigma^2} \sum_{n=1}^{N} \mathfrak{F}(y/\underline{\xi})\, \mathfrak{F}'(y/\underline{\xi}) \,. \tag{2.5}$$

The corresponding result for the continuous time case of Eq. (2.1) is

$$B = \frac{1}{\sigma^2} \int_{0}^{T} \mathfrak{F}(y/\underline{\xi})\, \mathfrak{F}'(y/\underline{\xi})\, dt. \tag{2.6}$$

Now consider an alternate approach to investigating the errors associated with nonlinear parameter estimation problems. A Taylor series expansion of $y(t, \underline{\xi})$ in terms of the elements of $\underline{\xi}$ about $\underline{\xi}^o$ gives,

$$y(t, \underline{\xi}) = y(t, \underline{\xi}^o) + \mathfrak{F}'(y/\underline{\xi})\Delta\underline{\xi} + \text{Remainder} \tag{2.7}$$

where

$$\Delta\underline{\xi} = \underline{\xi} - \underline{\xi}^o.$$

Assume we can make an estimate, $\hat{\underline{\xi}}$, from our observations and *apriori* information such that $\hat{\underline{\xi}} - \underline{\xi}^o$ is small enough to make the remainder in Eq. (2.7) negligible. Then the error analysis of our original nonlinear problem is equivalent to an error analysis of the linear parameter estimation model

$$z(t) = \mathfrak{F}'(y/\underline{\xi})\Delta\underline{\xi} + \eta(t) \tag{2.8}$$

where $\mathfrak{F}(y/\underline{\xi})$ is a known function of time and $\Delta\underline{\xi}$ is to be estimated. If maximum likelihood or, in this case, unbiased least squares estimates of $\Delta\xi$ are made, the resulting error covariance matrix, Σ, is given by the well-known result

$$\Sigma = B^{-1} \tag{2.9}$$

(For the discrete time case, see discussions on linear regression and the linear hypothesis in Ref. 3 through 6.)

The matrix B is called the information matrix* (or Fisher Information matrix, see Ref. 3). If we define

$$B(t) = \mathfrak{F}(y/\underline{\xi})\mathfrak{F}'(y/\underline{\xi}) \tag{2.10}$$

as the information matrix for a single observation at time t, then Eqs. (2.6) or (2.5)

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* It is also called the normal matrix.

shows that, for independent observation errors, the information matrices add. Thus if we make two sets of observations with information matrices B_1 and B_2, and if the two observation noise processes are mutually independent, then the total information matrix B, is simply

$$B = B_1 + B_2. \tag{2.11}$$

Often knowledge of the value of $\underline{\xi}^o$ is obtained from sources other than just the observations. If we have such *apriori* information, it can often be modeled as a set of values

$$\underline{z}_{apriori} = \underline{\xi}^o + \underline{\eta}_{apriori} \tag{2.12}$$

where $\underline{\eta}_{apriori}$ is a zero mean, Gaussian random vector with covariance matrix $B^{-1}_{apriori}$. If $\underline{\eta}_{apriori}$ is independent of the observation noise, the additive property of information makes $B + B_{apriori}$ the information matrix corresponding to both the *apriori* information and the observations.

Our results are not completely tied to a Gaussian assumption. Using the Taylor expansion approach of Eq. (2.7), the covariance matrix of Eq. (2.9) is that obtained by least squares unbiased estimators (minimum variance linear estimators) for any "white" but non-Gaussian $\eta(t)$. However, for simplicity, we continue to employ the Gaussian assumptions.

Henceforth we call the matrix Σ defined by Eq. (2.9) and Eq. (2.6) the covariance matrix of the estimate errors. Our preceding discussions show this to be a valid statement provided 1) we use maximum likelihood (matched filter) estimation techniques and 2) the signal-to-noise ratio and amount of *apriori* information are large enough to give small errors in the estimates of the parameters. If either condition is violated, our results are actually a bound on the obtainable accuracy.

3. SINGLE TARGET INFORMATION MATRIX

The formulae of Sec. 2 are now applied to a general model for a single target. A radar transmits a modulated sine wave with carrier frequency ω_c and time duration T. Let r(t) denote the range between the radar and a moving point reflector at time t. Let $\delta(t)$ denote the time delay between transmission and receipt of the signal; to be explicit, a signal received at time t, was transmitted at time $t-\delta(t)$. r(t) and $\delta(t)$ are related by

$$c\delta(t) = 2r\left(t - \frac{\delta(t)}{2}\right) \tag{3.1}$$

where c is the speed of light. Assume the equations of motion of the reflector are known to within the values of the unknown parameters, $\theta_1, \cdots, \theta_\rho$. Let $\underline{\theta}$ be the column vector

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_\rho \end{bmatrix} \tag{3.2}$$

The time delay at time t is written as $\delta(t, \underline{\theta})$. We do not specialize our results to any specific form for $\delta(t, \underline{\theta})$. However, a common model for accelerating targets is

$$\delta(t, \underline{\theta}) = \theta_1 + \theta_2(t-\tau) + \frac{\theta_3}{2}(t-\tau)^2 \tag{3.3}$$

where τ is some chosen reference time. For small accelerations, θ_1, θ_2 and θ_3 are approximately proportional to the target's range, range rate and range acceleration at time τ.

Let α denote the amplitude and β the phase of the reflected signal as received at the transmitter. Both α and β are considered unknown parameters. Let $\underline{\xi}$ denote the $\rho + 2$ dimensional column vector containing the elements of $\underline{\theta}$ and α and β. Let $\underline{\xi}^0$

and thus $\underline{\theta}^{o}$, α^{o} and β^{o} denote the true but unknown values of these parameters.

The carrier is modulated by both amplitude and frequency modulation. Let v(t) denote the amplitude modulation and let u(t) denote the frequency modulation.

Assume both quadrature components, $z_k(t)$ $k = 1, 2$, of the received signal are observed in the presence of additive, broad band, zero mean Gaussian noise. Then*

$$z_k(t) = y_k(t, \underline{\xi}^{o}) + \eta_k(t) \qquad k = 1, 2 \qquad 0 \le t \le T \tag{3.4}$$

where $\eta_1(t)$ and $\eta_2(t)$ are zero mean, stationary, white, mutually independent, Gaussian stochastic processes with two-sided noise power, σ^2 and

$$y_k(t, \underline{\xi}) = \alpha v(t, \underline{\theta}) \cos[\omega(t, \underline{\theta}) + \beta - \frac{\pi(k-1)}{2}] \quad k = 1, 2. \tag{3.5}$$

The exact form of $v(t, \underline{\theta})$ and $\omega(t, \underline{\theta})$ depends on the explicit problem. If both phase and envelope characteristics of the observed signal are used to form the desired estimates,

$$v(t, \underline{\theta}) = v[t - \delta(t, \underline{\theta})] \tag{3.6}$$

$$\omega(t, \underline{\theta}) = \int_0^{t-\delta(t, \underline{\theta})} [\omega_c + u(\tau)] d\tau \, . \tag{3.7}$$

In Sec. 5, the above and forms corresponding to other cases will be discussed.

Equations (3.4) and (3.5) define a nonlinear parameter estimation model of the type discussed in Sec. 2. Therefore, $\Sigma_{\underline{\xi}}$, the covariance matrix of the errors made in estimating $\underline{\xi}^{o}$ from observing $z_k(t)$ $0 \le t \le T$ is given in the sense of Sec. 2 by

* The time interval for which $y_k(t, \underline{\xi}^{o})$ is nonzero is really $t_1(\underline{\theta}^{o}) \le t \le T + \delta(T, \underline{\theta}^{o})$ where $t_1(\underline{\theta}^{o})$ is the first time for which $t - \delta(t, \underline{\theta}^{o})$ is nonnegative. However, for most problems of interest, an interval $0 \le t \le T$ is a satisfactory model for analysis. If not the necessary modifications are obvious.

$$\Sigma_{\underline{\xi}} = (B_{\underline{\xi}} + B_{\underline{\xi}, a\ priori})^{-1} \tag{3.8}$$

$$B_{\underline{\xi}} = \int_0^T B_{\underline{\xi}}(t)dt \tag{3.9}$$

$$B_{\underline{\xi}}(t) = B_{1,\underline{\xi}}(t) + B_{2,\underline{\xi}}(t) \tag{3.10}$$

$$B_{k,\underline{\xi}}(t) = \frac{1}{\sigma^2}\, \mathfrak{F}(y_k/\underline{\xi})\mathfrak{F}'(y_k/\underline{\xi}) \qquad k = 1, 2 \tag{3.11}$$

where $\mathfrak{F}(y_k/\underline{\xi})$ denotes the column vector of partial derivatives of $y_k(t, \underline{\xi})$ with respect to the components of $\underline{\xi}$, evaluated at $\underline{\xi} = \underline{\xi}^o$. $B_{\underline{\xi}}(t)$ is the information obtained at time t from observing both quadrature components of the received signal while $B_{\underline{\xi}}$ is the total information obtained over the time interval $0 \leq t \leq T$. $B_{\underline{\xi}, a\ priori}$ is the *a priori* information. Our final interest is in $\Sigma_{\underline{\xi}}$ but we often confine discussions to just $B_{\underline{\xi}}(t)$ as it is the basic quantity.

If we partition $\underline{\xi}$ as

$$\underline{\xi} = \begin{bmatrix} \alpha \\ \beta \\ \hdashline \underline{\theta} \end{bmatrix} \tag{3.12}$$

then

$$\mathfrak{F}(y_k/\underline{\xi}) = \begin{bmatrix} \mathfrak{F}(y_k/\alpha) \\ \mathfrak{F}(y_k/\beta) \\ \hdashline \mathfrak{F}(y_k/\underline{\theta}) \end{bmatrix} \tag{3.13}$$

where $\mathfrak{F}(y_k/\alpha)$ and $\mathfrak{F}(y_k/\beta)$ are scalars while $\mathfrak{F}(y_k/\underline{\theta})$ is a ρ-dimensional column vector. Here and in the rest of the report, we use dotted lines to indicate the partitioning

of vectors and matrices (which is not always as done in Eqs. (3.12) and (3.13). Define

$$\Omega_k = \omega(t,\underline{\theta}) + \beta - (k-1)\pi/2. \tag{3.14}$$

Then from Eq. (3.5)

$$\mathfrak{F}(y_k/\alpha) = v(t,\underline{\theta}) \cos \Omega_k \tag{3.15}$$

$$\mathfrak{F}(y_k/\beta) = -\alpha v(t,\underline{\theta}) \sin \Omega_k \tag{3.16}$$

$$\mathfrak{F}(y_k/\underline{\theta}) = \alpha[\mathfrak{F}(v/\underline{\theta}) \cos \Omega_k - v(t,\underline{\theta})\mathfrak{F}(\omega/\underline{\theta})\sin \Omega_k]. \tag{3.17}$$

Combining Eqs. (3.10) and (3.11) with Eqs. (3.15), (3.16) and (3.17) gives after manipulation and cancellation

$$B_{\underline{\xi}}(t) = \frac{\alpha^2}{\sigma^2}\left[\begin{array}{cc|c} \frac{v^2(t)}{\alpha^2} & 0 & \frac{1}{\alpha}v(t)\mathfrak{F}'(v/\underline{\theta}) \\ & v^2(t) & v^2(t)\mathfrak{F}'(\omega/\underline{\theta}) \\ \hline & & \mathfrak{F}(v/\underline{\theta})\mathfrak{F}'(v/\underline{\theta})+v^2(t)\mathfrak{F}(\omega/\underline{\theta})\mathfrak{F}'(\omega/\underline{\theta}) \end{array}\right] \tag{3.18}$$

where Eq. (3.18) is partitioned corresponding to Eq. (3.12) with the lower right hand element a $\rho \times \rho$ matrix. We indicate only half of the off-diagonal terms as $B_{\underline{\xi}}(t)$ is a symmetric matrix. Equation (3.18) is to be evaluated for $\underline{\xi} = \underline{\xi}^o$ (i.e., $\alpha = \alpha^o$, $\underline{\theta} = \underline{\theta}^o$) but for simplicity we do not include this or the dependence of v(t) on $\underline{\theta}$ in our notation.

In the following sections we work solely with the time delay $\delta(t,\underline{\theta})$ and in particular, $\mathfrak{F}(\delta/\underline{\theta})$. If the corresponding range, $r(t,\underline{\theta})$ is given instead, Eq. (3.1) must be used. A useful formulae obtained from Eq. (3.1) is

$$[c + \dot{r}(t - \delta(t,\underline{\theta}),\underline{\theta})]\mathfrak{F}(\delta/\underline{\theta}) = 2\mathfrak{F}(r/\underline{\theta})$$

where

$$\dot{r}(t - \delta(t,\underline{\theta}),\underline{\theta}) = \frac{dr(\tau,\underline{\theta})}{d\tau}\Bigg/_{\tau = t - \delta(t,\underline{\theta})} .$$

4. EFFECTS OF UNKNOWN PHASE AND AMPLITUDE ON TARGET PARAMETERS

Our analysis includes the variances of estimates of the phase, β and the amplitude α. These are sometimes of interest; for example, the variance of an estimate of α is a measure of detection ability. However, for many accuracy studies α and β are nuisance parameters and we are really interested in $\Sigma_{\underline{\theta}} = B_{\underline{\theta}}^{-1}$, the covariance matrix of the errors made in estimating the target parameters $\underline{\theta}$. $B_{\underline{\theta}}$ can be obtained by partitioning $B_{\underline{\xi}}$ as*

$$B_{\underline{\xi}} = \left[\begin{array}{c|c} B^{11} & B^{12} \\ \hline & B^{22} \end{array}\right] \tag{4.1}$$

and

$$\Sigma_{\underline{\xi}} = B_{\underline{\xi}}^{-1} = \left[\begin{array}{c|c} \Sigma^{11} & \Sigma^{12} \\ \hline & \Sigma^{22} \end{array}\right] . \tag{4.2}$$

If the partitioning is done as indicated by the dotted lines in Eq. (3.18),

$$\Sigma^{22} = \Sigma_{\underline{\theta}} = B_{\underline{\theta}}^{-1} . \tag{4.3}$$

The formulae for inverting partitioned matrices give

$$(\Sigma^{22})^{-1} = B^{22} - B^{21} (B^{11})^{-1} B^{12} . \tag{4.4}$$

Applying Eq. (4.4) to Eq. (3.18) gives

*If *a priori* information is present, we partition $B_{\underline{\xi}} + B_{\underline{\xi}, a\,priori}$.

$$\frac{\sigma^2}{\alpha^2} B_{\underline{\theta}} = \int_0^T \left[\mathfrak{F}(v/\underline{\theta})\, \mathfrak{F}'(v/\underline{\theta}) + v^2(t)\, \mathfrak{F}(\omega/\underline{\theta})\, \mathfrak{F}'(\omega/\underline{\theta}) \right] dt$$

$$- \frac{1}{\int_0^T v^2(t)\, dt} \left[G_v G_v' + G_\omega G_\omega' \right] \tag{4.5}$$

where

$$G_v = \int_0^T v(t)\, \mathfrak{F}(v/\underline{\theta})\, dt \tag{4.6}$$

$$G_\omega = \int_0^T v^2(t) \mathfrak{F}(\omega/\underline{\theta})\, dt \quad . \tag{4.7}$$

The matrix, $G_v G_v' + G_\omega G_\omega'$ is nonnegative definite and measures the loss in information about $\underline{\theta}$ that results from an unknown phase and amplitude. As we shall see, this information loss is zero in some special cases.

We do not use $B_{\underline{\theta}}$ in our further development as consideration of the full $B_{\underline{\zeta}}$ matrix leads to much simpler results and a better understanding of the problem. However, Eq. (4. 4) can be employed at any point B_θ is of interest.

5. SINGLE TARGET EQUIVALENT LINEAR MODELS

We now discuss linear equivalent models for three cases of a single target:

R-F Phase Used for all Parameters

R-F Phase Used for Motion Parameters

R-F Phase Not Used.

These cases do not exhaust the range of possibilities but are sufficiently illustrative to indicate how other situations are handled. The choice of which case to use naturally depends on the problem under investigation; in particular, the type of the equipment available and the nature of the ambiguity problem.

In Sec. 2, we discussed the linearization of a nonlinear problem by truncation of a Taylor series expansion. In this sense, we could form an equivalent linear model for our radar problem by using the partial derivatives of Eqs. (3.15), (3.16) and (3.17). However, we follow a more productive course and hypothesize two sets of observations

$$z_a(t) = \underline{\varphi}_a'(t)\underline{\xi}^o + \eta_a(t) \tag{5.1}$$

$$z_f(t) = \underline{\varphi}_f'(t)\underline{\xi}^o + \eta_f(t) \tag{5.2}$$

where $\eta_a(t)$ and $\eta_f(t)$ are zero mean, stationary, white, mutually independent Gaussian stochastic processes with two-sided noise power σ^2, $\underline{\xi}^o$ is the parameter vector to be estimated, and $\underline{\varphi}_a(t)$ and $\underline{\varphi}_f(t)$ are column vector functions of time. We know from Sec. 2 (or Eqs. (3.8) through (3.11)) that the information obtained from observing $z_a(t)$ and $z_f(t)$ at time t is,

$$B_{\underline{\xi}}(t) = B_{a,\underline{\xi}}(t) + B_{f,\underline{\xi}}(t) \tag{5.3}$$

where

$$B_{a,\underline{\xi}}(t) = \frac{1}{\sigma^2}\,\underline{\varphi}_a(t)\,\underline{\varphi}_a'(t)$$

$$B_{f,\underline{\xi}}(t) = \frac{1}{\sigma^2}\,\underline{\varphi}_f(t)\,\underline{\varphi}_f'(t) \quad .$$

Equations (5.1) and (5.2) are an equivalent linear model for a radar problem when the vector functions $\underline{\varphi}_a(t)$ and $\underline{\varphi}_f(t)$ are chosen to make $B_{\underline{\xi}}(t)$ of Eq. (5.3) the same as the $B_{\underline{\xi}}(t)$ of the original radar problem.

In a certain sense, we choose $\underline{\varphi}_a(t)$ and $\underline{\varphi}_f(t)$ such that $B_{a,\underline{\xi}}(t)$ represents the information obtained from the amplitude modulation while $B_{f,\underline{\xi}}(t)$ represents the information obtained from the frequency modulation and the R-F phase. As will be seen, this dichotomy is not precise as $\underline{\varphi}_f(t)$ is dependent on the amplitude modulation v(t) as well as the frequency modulation, u(t). However for many situations, v(t) itself determines the energy content of the signal while it is the structure of $\frac{d}{dt}v(t)$ that determines the effectiveness of the amplitude modulation. Admittedly, this point of view is not universally valid but in the author's opinion, it is of sufficient value to call $\underline{\varphi}_a(t)$ and $\underline{\varphi}_f(t)$ equivalent models for amplitude and frequency respectively.

If any *a priori* information exists, the equivalent linear model of Eqs. (5.1) and (5.2) is supplemented by the additional observations (see Eq. (2.12)

$$\underline{z}_{a\,priori} = \underline{\xi}^o + \underline{\eta}_{a\,priori} \tag{5.4}$$

with $B_{\underline{\xi},a\,priori}$ the corresponding information. For example, if $\Sigma_{\underline{\theta},a\,priori}$ denotes the covariance matrix of an estimate of $\underline{\theta}$ made from previous tracking of the target and if we partition as

$$\underline{\xi} = \begin{bmatrix} \alpha \\ \beta \\ \hline \underline{\theta} \end{bmatrix} \tag{5.5}$$

then*

---

* Note that $B^{-1}_{\underline{\xi},a\,priori}$ does not exist. In the same sense, $\Sigma_{\underline{\theta},a\,priori}$ does not have to exist.

$$
B_{\underline{\xi}, a\,priori} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline & & \Sigma^{-1}_{\underline{\theta},\, a\,priori} \end{array}\right]
$$

Previous tracking might also provide useful information on the target cross section which implies *a priori* information on the amplitude α. *A priori* information can also arise from other sources such as equipment which maintains a measure of the phase. If $\sigma^2_{\beta, a\,priori}$ denotes the variance of these equipment errors and other phase uncertainties, then

$$
B_{\underline{\ }, a\,priori} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ & \frac{1}{\sigma^2_{\beta, a\,priori}} & 0 \\ \hline & & 0 \end{array}\right] \tag{5.6}
$$

is the corresponding *a priori* information.

The limiting case where the carrier frequency, ω_c, is much larger than the effective signal bandwidth is of special interest. This can be obtained by simply dropping the lower order terms in ω_c that affect the information on $\underline{\theta}$. It is difficult to specify the range of carrier frequency to bandwidth for which this approximation is useful as, in general, the nature of the target's motion is also a contributing factor. However, the large ω_c approximation is very useful for a wide range of problems. The large ω_c case can also be obtained by appropriately modifying the definitions of $v(t, \underline{\theta})$ and $\omega(t, \underline{\theta})$.

To illustrate the equivalent linear model, we begin with the general case. It is seen by inspection, that for

$$
\underline{\varphi}_a(t) = \alpha \left[\begin{array}{c} \frac{v(t)}{\alpha} \\ 0 \\ \hline \mathfrak{F}(\underline{v}/\underline{\theta}) \end{array}\right]
$$

$$
\underline{\varphi}_f(t) = \alpha v(t) \left[\begin{array}{c} 0 \\ 1 \\ \hline \mathfrak{F}(\omega/\underline{\theta}) \end{array}\right]
$$

the $B_{\underline{\xi}}(t)$ obtained from Eq. (5.3) is equal to the $B_{\underline{\xi}}(t)$ of Eq. (3.18).

We now consider three special cases and obtain explicit formulae for $\mathfrak{F}(\omega/\underline{\theta})$ and $\mathfrak{F}(v/\underline{\theta})$.

R-F Phase Used For All Parameters

For the case when the R-F phase is used for all parameters, the equations for $v(t,\underline{\theta})$ and $\omega(t,\underline{\theta})$ are given by Eq. (3.6) and (3.7). The resulting partial derivatives are

$$\mathfrak{F}(v/\underline{\theta}) = -\dot{v}(t)\,\mathfrak{F}(\delta/\underline{\theta}) \tag{5.7}$$

$$\mathfrak{F}(\omega/\underline{\theta}) = -[\omega_c + u(t)]\,\mathfrak{F}(\delta/\underline{\theta}) \tag{5.8}$$

where

$$\dot{v}(t) = \frac{dv(t)}{dt} \quad .$$

The time functions of Eqs. (5.7) and (5.8) are actually evaluated at time $t-\delta(t,\underline{\theta}^o)$ but evaluation at time t is satisfactory for most cases. Therefore we simplify our notation by dropping the $\underline{\theta}^o$ dependence. Evaluation of Eq. (3.18) for Eqs. (5.7) and (5.8) gives

$$B_{\underline{\xi}}(t) = \frac{\alpha^2}{\sigma^2}\left[\begin{array}{cc|c} \dfrac{v^2(t)}{\alpha^2} & 0 & \dfrac{-\dot{v}(t)\,\mathfrak{F}'(\delta/\underline{\theta})}{\alpha} \\ & v^2(t) & v^2(t)[\omega_c + u(t)]\,\mathfrak{F}'(\delta/\underline{\theta}) \\ \hline & & \left\{\dot{v}^2(t) + v^2(t)[\omega_c + u(t)]^2\right\}\mathfrak{F}(\delta/\underline{\theta})\,\mathfrak{F}'(\delta/\underline{\theta}) \end{array}\right] \tag{5.9}$$

where the partitioning is done as in Eq. (5.5). By inspection we see that

$$\underline{\varphi}_a(t) = \alpha \left[\begin{array}{c} \dfrac{v(t)}{\alpha} \\ \\ 0 \\ \hdashline \\ \dot{v}(t)\,\mathfrak{F}(\delta/\underline{\theta}) \end{array} \right] \tag{5.10}$$

$$\underline{\varphi}_f(t) = \alpha v(t) \left[\begin{array}{c} 0 \\ \\ 1 \\ \hdashline \\ [\omega_c + u(t)]\,\mathfrak{F}(\delta/\underline{\theta}) \end{array} \right] \tag{5.11}$$

combine with Eq. (5.3) to give the same $B_\xi(t)$ as that of Eq. (5.9). When the R-F phase is used for all parameters, *apriori* information on the phase is essential and this is modeled as in Eq. (5.6). Thus our equivalent linear system consists of Eqs. (5.1) and (5.2) with the $\underline{\varphi}_a(t)$ and $\underline{\varphi}_f(t)$ of Eqs. (5.10) and (5.11), combined with the *apriori* model of Eqs. (5.4) and (5.6).

If the phase β is known perfectly, i.e., $\sigma^2_{\beta,apriori} = 0$, the row and column of Eq. (5.9) corresponding to β are simply deleted. If in addition we use the large ω_c approximation, our equivalent linear model for estimating $\underline{\theta}$ is simply

$$\begin{aligned} z_f(t) &= \underline{\varphi}_f'(t)\,\underline{\xi}^o + \eta_f(t) \\ \underline{\varphi}_f(t) &= \alpha v(t)\omega_c\,\mathfrak{F}(\delta/\underline{\theta}) \end{aligned} \tag{5.12}$$

where

$$\underline{\xi} = \underline{\theta} \quad .$$

R-F Phase Used For Motion Parameters

Consider the case

$$\delta(t,\underline{\theta}) = \theta_1 + \delta_m(t,\underline{\theta}_m) \tag{5.13}$$

where

$$\theta_1 = \delta(t,\underline{\theta})/_{t=\tau} \tag{5.14}$$

and $\underline{\theta}_m$ is the ρ-1 dimensional column vector containing all elements of $\underline{\theta}$ except θ_1. θ_1 is the time delay at time τ and $\delta_m(t,\underline{\theta}_m)$ describes the change in $\delta(t,\underline{\theta})$ with the motion of the target. Thus the elements of $\underline{\theta}_m$ are the parameters of the target's motion. For the accelerating target example of Eq. (3. 3),

$$\delta_m(t,\underline{\theta}_m) = \theta_2(t-\tau) + \frac{\theta_3(t-\tau)^2}{2}$$

$$\underline{\theta}_m = \begin{bmatrix} \theta_2 \\ \\ \theta_3 \end{bmatrix} .$$

In many situations, lack of knowledge of phase β or more often the ambiguity problem prevent the effective use of the R-F phase in the determination of θ_1 although it can be used for the target motion parameters $\underline{\theta}_m$. For such cases we write

$$v(t,\underline{\theta}) = v(t-\theta_1 - \delta_m(t,\underline{\theta}_m)) \tag{5.15}$$

$$\omega(t,\underline{\theta}) = \omega_c t - \omega_c \delta_m(t,\underline{\theta}_m) + \int_0^{t-\theta_1-\delta_m(t,\underline{\theta}_m)} u(s)ds. \tag{5.16}$$

In Eq. (5. 16) the $\omega_c\theta_1$ term has been combined with the phase shift β (see Eq. (3. 5)) ; that is, $\beta + \omega_c\theta_1$ is considered the unknown phase shift rather that just β. This

change of variables is made for convenience, not necessity. For example, if Eq. (5. 13) is substituted into Eq. (5. 9) and the partitioning technique of Sec. 4 applied to the resulting $B_{\underline{\xi}}$, the same $\Sigma_{\underline{\theta}}$ we obtain by the change of variables results as many terms cancel. However, if we keep only the highest order terms in ω_c, the $B_{\underline{\xi}}$ obtained from Eq. (5. 9) becomes singular for $\delta(t, \underline{\theta})$ of Eq. (5. 13). The change of variables side tracks these unnecessary complications.

If we partition $\underline{\theta}$ as

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \hline \underline{\theta}_m \end{bmatrix}$$

then from Eqs. (5. 15) and (5. 16)

$$\mathfrak{F}(v/\underline{\theta}) = -\dot{v}(t) \begin{bmatrix} 1 \\ \hline \mathfrak{F}(\delta_m/\underline{\theta}_m) \end{bmatrix} \tag{5. 17}$$

$$\mathfrak{F}(\omega/\underline{\theta}) = - \begin{bmatrix} u(t) \\ \hline [\omega_c + u(t)]\,\mathfrak{F}(\delta_m/\underline{\theta}_m) \end{bmatrix} \tag{5. 18}$$

The resulting $B_{\underline{\xi}}(t)$ of Eq. (3. 18) is*

* A constant frequency modulation $u(t) = k$, is equivalent to a carrier frequency of $\omega_c + k$ with $u(t) = 0$. This obvious physical deduction is not evident from Eq. (5. 19) but use of the matrix partitioning of Sec. 4 shows it to be true as various terms cancel.

$$
B_{\underline{\zeta}}(t) = \frac{\alpha^2}{\sigma^2}\left[\begin{array}{ccc|c}
\frac{1}{\alpha^2}v^2(t) & 0 & -\frac{1}{\alpha}v(t)\dot{v}(t) & -\frac{1}{\alpha}v(t)\dot{v}(t)\mathfrak{F}'(\delta_m/\underline{\theta}_m) \\
 & v^2(t) & -v^2(t)u(t) & -\{v^2(t)[\omega_c + u(t)]\}\mathfrak{F}'(\delta_m/\underline{\theta}_m) \\
 & & \dot{v}^2(t)+v^2(t)u^2(t) & \{\dot{v}^2(t)+v^2(t)u(t)[\omega_c+u(t)]\}\mathfrak{F}'(\delta_m/\underline{\theta}_m) \\
\hline
 & & & \{\dot{v}^2(t)+v^2(t)[\omega_c+u(t)]^2\}\mathfrak{F}(\delta_m/\underline{\theta}_m)\mathfrak{F}'(\delta_m/\underline{\theta}_m)
\end{array}\right] \tag{5.19}
$$

where we have partitioned as

$$
\underline{\zeta} = \left[\begin{array}{c}
\alpha \\
\beta \\
\theta_1 \\
\hline
\underline{\theta}_m
\end{array}\right]
$$

By inspection we see that the corresponding forms of Eqs. (5. 1) and (5. 2) are

$$
\underline{\varphi}_a(t) = \alpha\left[\begin{array}{c}
\frac{1}{\alpha}v(t) \\
0 \\
\dot{v}(t) \\
\hline
\dot{v}(t)\,\mathfrak{F}(\delta_m/\underline{\theta}_m)
\end{array}\right]
$$

$$\underline{\varphi}_f(t) = \alpha v(t) \left[\begin{array}{c} 0 \\ 1 \\ u(t) \\ \hline [\omega_c + u(t)]\,\mathfrak{F}(\delta_m/\underline{\theta}_m) \end{array} \right]$$

For the large ω_c approximation,

$$\underline{\varphi}_a(t) = \alpha \left[\begin{array}{c} \frac{1}{\alpha} v(t) \\ 0 \\ \dot{v}(t) \\ \hline 0 \end{array} \right]$$

$$\underline{\varphi}_f(t) = \alpha v(t) \left[\begin{array}{c} 0 \\ 1 \\ u(t) \\ \hline \omega_c\,\mathfrak{F}(\delta_m/\underline{\theta}_m) \end{array} \right] .$$

As mentioned earlier,the large ω_c case can also be derived directly by changing the definitions of $v(t,\underline{\theta})$ and $\omega(t,\underline{\theta})$. For example, the present case can also be obtained using

$$v(t,\underline{\theta}) = v(t - \theta_1)$$

$$\omega(t,\underline{\theta}) = \omega_c t - \omega_c \delta_m(t,\underline{\theta}_m) + \int_0^{t-\theta_1} u(s)ds$$

and substituting the resulting $\mathfrak{F}(v/\underline{\theta})$ and $\mathfrak{F}(\omega/\underline{\theta})$ directly into Eq. (3.18). Reference 2 uses this approach.

R-F Phase Not Used

If the R-F phase is not used at all, the phase parameter β is dropped from our model by removing the corresponding row and column of Eq. (3.18). Using

$$v(t,\underline{\theta}) = v[t-\delta(t,\underline{\theta})]$$

$$\omega(t,\underline{\theta}) = \omega_c t + \int_0^{t-\delta(t,\underline{\theta})} u(\tau)d\tau$$

and partitioning as

$$\underline{\xi} = \begin{bmatrix} \alpha \\ \hline \underline{\theta} \end{bmatrix}$$

we obtain

$$B_{\underline{\xi}}(t) = \frac{\alpha^2}{\sigma^2} \left[\begin{array}{c|c} \dfrac{v^2(t)}{\alpha^2} & -\dfrac{1}{\alpha} v(t)\dot{v}(t)\, \mathfrak{F}'(\delta/\underline{\theta}) \\ \hline & [\dot{v}^2(t) + v^2(t)\,u^2(t)]\mathfrak{F}(\delta/\underline{\theta})\, \mathfrak{F}'(\delta/\underline{\theta}) \end{array}\right] . \qquad (5.20)$$

The corresponding linear model is

$$\underline{\varphi}_a(t) = \alpha \left[\begin{array}{c} \dfrac{v(t)}{\alpha} \\ \hdashline \dot{v}(t)\mathfrak{F}(\delta/\underline{\theta}) \end{array} \right]$$

$$\underline{\varphi}_f(t) = \alpha v(t) \left[\begin{array}{c} 0 \\ \hdashline u(t)\mathfrak{F}(\delta/\underline{\theta}) \end{array} \right] .$$

6. MULTIPLE TARGETS

In many applications, target resolution rather than single target accuracy is of prime interest. Target resolution refers to the ability of the radar to determine (resolve) the properties of separate targets when it observes the sum of the reflections from each target. One approach to this problem is the inspection of the ambiguity function. However, a more precise technique is to simply extend the single target analysis to the multiple target case and thereby investigate the following question: Given a noise corrupted observation of the sum of the signals reflected from several targets, how well can the parameters of all the targets be estimated?* (Reference 7 uses this technique to investigate angular resolution.)

For simplicity, we restrict our explicit equations to the case of just two targets. The general multiple target case is conceptually a trivial extension but the notational complications tend to obscure the basic ideas. In the two-target case, there are $2(\rho+2)$ unknown parameters which we denote by the column vector

$$\underline{Z} = \begin{bmatrix} \underline{\zeta}_1 \\ \hline \underline{\zeta}_2 \end{bmatrix} \tag{6.1}$$

where

$$\underline{\zeta}_j = \begin{bmatrix} \alpha_j \\ \beta_j \\ \hline \underline{\theta}_j \end{bmatrix} \qquad j = 1, 2 \tag{6.2}$$

and of course, α_j, β_j and $\underline{\theta}_j$ denote the amplitude, phase and range parameters of the j^{th} target.

---

* Our single target results can be interpreted as the curvature of the ambiguity function at the true values of the parameters (see Ref. 2). Thus the multiple target analysis can be considered the curvature of a higher dimensional ambiguity function.

For the two-target case, the two quadrature components corresponding to Eq. (3. 4) are given by

$$z_k(t) = Y_k(t, \underline{Z}^o) + \eta_k(t) \qquad k = 1, 2$$

where

$$Y_k(t, \underline{Z}) = y_k(t, \underline{\xi}_1) + y_k(t, \underline{\xi}_2)$$

and $y_k(t, \underline{\xi})$ is given by Eq. (3. 5). It follows that, partitioning as in Eq. (6. 1),

$$\mathfrak{F}(Y_k/\underline{Z}) = \left[\begin{array}{c} \mathfrak{F}(y_k/\underline{\xi}_1) \\ \hline \mathfrak{F}(y_k/\underline{\xi}_2) \end{array}\right] \tag{6. 3}$$

where $\mathfrak{F}(y_k/\underline{\xi}_j)$ is the column vector of partial derivatives of $y_k(t, \underline{\xi}_j)$ with respect to the elements of $\underline{\xi}_j$. Using Eq. (3. 11) and (3. 10) we have

$$B_{\underline{Z}}(t) = \frac{1}{\sigma^2} \left\{ \mathfrak{F}(Y_1/\underline{Z})\mathfrak{F}'(Y_1/\underline{Z}) + \mathfrak{F}(Y_2/\underline{Z})\mathfrak{F}'(Y_2/\underline{Z}) \right\} .$$

Substituting Eq. (6. 3) gives

$$B_{\underline{Z}}(t) = \left[\begin{array}{c|c} B_{\underline{\xi}_1}(t) & C(t) \\ \hline & B_{\underline{\xi}_2}(t) \end{array}\right] \tag{6. 4}$$

where partitioning is as in Eq. (6. 1), $B_{\underline{\xi}_1}(t)$ and $B_{\underline{\xi}_2}(t)$ are the information matrices which would arise if the targets were observed separately, and

$$C(t) = \frac{1}{\sigma^2} \sum_{k=1}^{2} \mathfrak{F}(y_k/\underline{\xi}_1)\mathfrak{F}'(y_k/\underline{\xi}_2) \quad . \tag{6.5}$$

Using Eq. (3.13),

$$C(t) = \frac{1}{\sigma^2} \sum_{k=1}^{2} \left[\begin{array}{cc|c} \mathfrak{F}(y_k/\alpha_1)\mathfrak{F}(y_k/\alpha_2) & \mathfrak{F}(y_k/\alpha_1)\mathfrak{F}(y_k/\beta_2) & \mathfrak{F}(y_k/\alpha_1)\mathfrak{F}'(y_k/\underline{\theta}_2) \\ \mathfrak{F}(y_k/\beta_1)\mathfrak{F}(y_k/\alpha_2) & \mathfrak{F}(y_k/\beta_1)\mathfrak{F}(y_k/\beta_2) & \mathfrak{F}(y_k/\beta_1)\mathfrak{F}'(y_k/\underline{\theta}_2) \\ \hline \mathfrak{F}(y_k/\alpha_2)\mathfrak{F}(y_k/\underline{\theta}_1) & \mathfrak{F}(y_k/\beta_2)\mathfrak{F}(y_k/\underline{\theta}_1) & \mathfrak{F}(y_k/\underline{\theta}_1)\mathfrak{F}'(y_k/\underline{\theta}_2) \end{array} \right] \tag{6.6}$$

where partitioning is as in Eq. (6.2). The necessary partial derivatives are given by Eq. (3.15) through Eq. (3.17) with the obvious subscript additions. Define

$$\Delta(t) = \omega(t, \underline{\theta}_1) - \omega(t, \underline{\theta}_2) + \beta_1 - \beta_2 \quad . \tag{6.7}$$

For simplicity we do not indicate the dependence of $\Delta(t)$ on $\underline{\theta}_1, \underline{\theta}_2$, β_1 and β_2. Also define

$$v_j(t) = v(t, \underline{\theta}_j) \qquad j = 1, 2 \quad .$$

Substituting Eqs. (3.15), (3.16), (3.17) into Eq. (6.6) then gives after reduction by use of trigonometric identities,

$$
C(t) = \frac{1}{\sigma^2}
\left[
\begin{array}{cc|c}
v_1(t)v_2(t)\cos[\Delta(t)] & \alpha_2 v_1(t)v_2(t)\sin[\Delta(t)] & \alpha_2 v_1(t)\{\cos[\Delta(t)]\,\mathfrak{F}'(v/\underline{\theta}_2)+v_2(t)\sin[\Delta(t)]\,\mathfrak{F}'(\omega/\underline{\theta}_2)\} \\
-\alpha_1 v_1(t)v_2(t)\sin[\Delta(t)] & \alpha_1\alpha_2 v_1(t)v_2(t)\cos[\Delta(t)] & \alpha_1\alpha_2 v_1(t)\{-\sin(\Delta(t))\,\mathfrak{F}'(v/\underline{\theta}_2)+v_2(t)\cos[\Delta(t)]\,\mathfrak{F}'(\omega/\underline{\theta}_2)\} \\
\hline
\begin{array}{l}\alpha_1 v_2(t)\{\cos[\Delta(t)]\,\mathfrak{F}(v/\underline{\theta}_1) \\ \quad -v_1(t)\sin[\Delta(t)]\,\mathfrak{F}(\omega/\underline{\theta}_2)\}\end{array} & \begin{array}{l}\alpha_1\alpha_2 v_2(t)\{\sin[\Delta(t)]\,\mathfrak{F}(v/\underline{\theta}_1) \\ \quad +v_1(t)\cos[\Delta(t)\,\mathfrak{F}(\omega/\underline{\theta}_1)\}\end{array} & \begin{array}{l}\alpha_1\alpha_2\{\cos[\Delta(t)][\mathfrak{F}(v/\underline{\theta}_1)\,\mathfrak{F}'(v/\underline{\theta}_2)+v_1(t)v_2(t)\,\mathfrak{F}(\omega/\underline{\theta}_1)\,\mathfrak{F}'(\omega/\underline{\theta}_2)] \\ \quad +\sin[\Delta(t)][v_2(t)\,\mathfrak{F}(v/\underline{\theta}_1)\,\mathfrak{F}'(\omega/\underline{\theta}_2)-v_1(t)\,\mathfrak{F}(\omega/\underline{\theta}_1)\,\mathfrak{F}'(v/\underline{\theta}_2)]\}\end{array}
\end{array}
\right]
$$

Eq. (6. 8)

Thus the information matrix, $B_{\underline{Z}}(t)$ of Eq. (6. 4) is completely specified by Eqs. (6. 8) and (3. 18).

As in the single target case, we have

$$\Sigma_{\underline{Z}} = (B_{\underline{Z}} + B_{\underline{Z},\, a\,priori})^{-1}$$

$$B_{\underline{Z}} = \int_0^{T_t} B_{\underline{Z}}(t)dt$$

where $\Sigma_{\underline{Z}}$ is the covariance matrix of the errors in the unbiased estimate of $\underline{Z}$. In the multiple target case, the possible range differences spread out the time duration of observations and this is indicated by making the observation interval 0 to T_t where T_t is simply chosen large enough to encompass the actual observation interval.

One important property can be observed directly from Eq. (6. 4). Define, with partitioning as in Eq. (6. 1)

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline & A_2 \end{array}\right]$$

where with partitioning as in Eq. (6. 2)

$$A_j = \left[\begin{array}{cc|c} 1 & 0 & \\ & \alpha_j & \\ \hline & & \alpha_j I \end{array}\right] \quad j = 1, 2$$

where 1 is a ρ by ρ unit matrix. Eq. (6. 4) can be written as

$$B_{\underline{Z}}(t) = A \hat{B}_{\underline{Z}}(t) A$$

where $\hat{B}_{\underline{Z}}(t)$ does not depend on the amplitude parameters α_1 and α_2. Thus if there is no *apriori* information,

$$\Sigma_{\underline{Z}} = A^{-1} \left[\int_0^{T_t} \hat{B}_{\underline{Z}}(t)dt \right]^{-1} A^{-1}$$

which shows that the amplitude (cross section), α_2, of the second target has no effect on the accuracy with which the parameters of the first target can be estimated and vice versa.* (See also the angular resolution problem of Ref. 7.) This property is a result of our assumption that observation noise is the only source of error. In practice, uncertainties in waveform shape and imperfections in the data processing equipment will introduce other errors which, for large cross section targets, may not be negligible.

Now let us consider an equivalent linear model for the two-target case. It is convenient to resort to a vector formulation. Assume the q_1-dimensional column vector $\underline{z}(t)$ is observed where

$$\underline{z}(t) = \Phi'(t)\underline{Z}^{o} + \underline{\eta}(t) \qquad 0 \le t \le T_t \tag{6.9}$$

where $\underline{Z}^{o}$ is a q_2-dimensional column vector of unknown parameters, $\Phi(t)$ is a q_1 by q_2 matrix of known time functions and $\underline{\eta}(t)$ is a q_1 dimensional, zero mean white Gaussian noise process with

$$E[\underline{\eta}(t)\underline{\eta}'(t-\tau)] = \delta(\tau)\Sigma_{\underline{\eta}} \tag{6.10}$$

where in Eq. (6.10), $\delta(\tau)$ is the Dirac delta function and not a time delay. For the model of Eq. (6.9) and (6.10) we have, (for the corresponding discrete time case, see Ref. 6)

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* Unless, of course, α_2 is known to be zero.

$$\Sigma_{\underline{Z}} = (B_{\underline{Z}} + B_{\underline{Z}, a\,priori})^{-1} \tag{6.11}$$

$$B_{\underline{Z}} = \int_0^{T_t} B_{\underline{Z}}(t)dt \tag{6.12}$$

$$B_{\underline{Z}}(t) = \Phi(t)\Sigma_{\underline{\eta}}^{-1}\Phi'(t) \tag{6.13}$$

The equivalent linear model for the single target, Eqs. (5.1) and (5.2), can be written in terms of this vector modei as

$$q_1 = 2$$

$$q_2 = \rho + 2$$

$$\underline{Z} = \underline{\xi}$$

$$\Phi'(t) = \begin{bmatrix} \underline{\varphi}_a'(t) \\ \\ \underline{\varphi}_f'(t) \end{bmatrix} \tag{6.14}$$

$$\underline{z}(t) = \begin{bmatrix} z_a(t) \\ \\ z_f(t) \end{bmatrix} \tag{6.15}$$

$$\underline{\eta}(t) = \begin{bmatrix} \eta_a(t) \\ \\ \eta_f(t) \end{bmatrix} \tag{6.16}$$

$$\Sigma_{\underline{\eta}} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.17}$$

For the two-target case, we define, by analogy with the single target case of Sec. 5,

$$\mathcal{L}_{a,j}(t) = \alpha_j \left[\begin{array}{c} \dfrac{v_j(t)}{\alpha_j} \\ 0 \\ \hdashline \mathfrak{F}(v/\underline{\theta}_j) \end{array}\right] \tag{6.18}$$

$$\mathcal{L}_{f,j}(t) = \alpha_j v_j(t) \left[\begin{array}{c} 0 \\ 1 \\ \hdashline \mathfrak{F}(\omega/\underline{\theta}_j) \end{array}\right] . \tag{6.19}$$

We further define

$$\Theta'(t) = \begin{bmatrix} \mathcal{L}'_{a,1}(t) & 0 \\ \mathcal{L}'_{f,1}(t) & 0 \\ 0 & \mathcal{L}'_{a,2}(t) \\ 0 & \mathcal{L}'_{f,2}(t) \end{bmatrix} \tag{6.20}$$

$$\Xi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cos\Delta(t) & -\sin\Delta(t) \\ \sin\Delta(t) & \cos\Delta(t) \end{bmatrix} \tag{6.21}$$

Then by algebraic manipulation, it can be shown that $B_{\underline{Z}}(t)$ of Eq. (6.4) can be written

$$B_{\underline{Z}}(t) = \frac{1}{\sigma^2}\, \Theta(t)\Xi(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Xi'(t)\Theta'(t) \ . \tag{6.22}$$

Comparison of Eq. (6.22) with Eq. (6.13) shows that two different linear equivalent models are of interest. For both models we have

$$q_2 = 2(\rho + 2)$$

$$\underline{Z} = \begin{bmatrix} \underline{\xi}_1 \\ \hdashline \underline{\xi}_2 \end{bmatrix} \ .$$

One possible linear equivalent model is then given by

$$q_1 = 2$$

$$\Sigma_\eta = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.23}$$

and

$$\Phi'(t) = \Xi'(t)\Theta'(t)$$

or

$$\Phi'(t) = \begin{bmatrix} \underline{\varphi}'_{a,1}(t) & \cos\Delta(t)\ \underline{\varphi}'_{a,2}(t) + \sin\Delta(t)\ \underline{\varphi}'_{f,2}(t) \\ \underline{\varphi}'_{f,1}(t) & -\sin\Delta(t)\ \underline{\varphi}'_{a,2}(t) + \cos\Delta(t)\ \underline{\varphi}'_{f,2}(t) \end{bmatrix} \ . \tag{6.24}$$

The second linear equivalent model is given by

$$q_1 = 4$$

$$\Phi'(t) = \Theta'(t) \tag{6.25}$$

and

$$\Sigma_{\underline{\eta}}^{-1} = \frac{1}{\sigma^2}\, \Xi(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Xi'(t)$$

or

$$\Sigma_{\underline{\eta}}^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & 0 & \cos\Delta(t) & \sin\Delta(t) \\ 0 & 1 & -\sin\Delta(t) & \cos\Delta(t) \\ \cos\Delta(t) & -\sin\Delta(t) & 1 & 0 \\ \sin\Delta(t) & \cos\Delta(t) & 0 & 1 \end{bmatrix} . \tag{6.26}$$

The equivalent linear model of Eq. (6.23) and (6.24) differs from the single target case due to the extra parameters and a more complicated expression for $\Phi'(t)$. The equivalent linear model of Eq. (6.25) and (6.26) employs four instead of two observations and has the following interesting property: If $\Sigma_{\underline{\eta}}^{-1}$ of Eq. (6.26) were $\frac{1}{\sigma^2}$ I, the model would correspond to the case wherein the reflections of the two targets are observed separately. Thus all the effects of the multiple target are contained in the cross correlation terms of Eq. (6.26). A disadvantage of the second model is that $\Sigma_{\underline{\eta}}^{-1}$ of Eq. (6.26) is a singular matrix; that is, $\Sigma_{\underline{\eta}}$ does not exist.

Since the multiple target model employs the same $\varphi_{\underline{a}}(t)$ and $\varphi_{\underline{f}}(t)$ used for the single target case, the discussions in Sec. 5 on the various special cases need not be repeated. The additional term, $\Delta(t)$, of Eq. (6.7) is easily found from the equations $\omega(t,\underline{\theta})$ given in Sec. 5.

In a multiple target environment, the question of how many targets are present is very important. For example, if two targets are assumed when actually there is only one, the estimates of the single target's parameters are degraded. The variances of

the estimates of the amplitude parameters, α_j, measure how well the number of targets actually present can be decided. (The corresponding discussion in Ref. 7 is more detailed.)

The matrix partitioning t[illegible] discussed in Sec. 4 is both applicable and useful in the multiple target case. For example, all the parameters of a second target might be considered nuisance parameters.

In many single target problems, the actual values of the unknown parameters do not affect the information matrix. However, for multiple targets the information matrix and hence the covariance matrix is strongly dependent on the relative values of the parameters of the different targets. Thus in order to obtain a true picture of target resolution ability, it is necessary to investigate the effects of changes in target separation. (Reference 7 indicates the type of results which may be expected.)

7. DISCUSSION

We have used the information matrix as a basis for investigating the accuracy and resolution ability of a wide class of radar signals. The inverse of this information matrix is a lower bound on the covariance matrix of the errors obtainable from any unbiased estimate of the unknown parameters. In theory, this bound is approached as the signal-to-noise ratio increases if maximum likelihood parameter estimation procedures are used. However, for very large signal-to-noise ratios, errors in factors such as waveform shape and data processing equipment become as important as those due to observation noise. Thus this bound is obtainable only for a limited range of signal-to-noise ratios. We have not attempted to discuss methods of implementing the required estimation procedures as there are many possibilities and the choice of technique depends very much on the explicit problem being investigated.

The inverse of the information matrix is usually of prime interest. In some special cases an analytic matrix inversion is practical (see for example Refs. 2 and 8) but there seems to be little hope for the general models we have considered. Thus if explicit closed form expressions are desired, the problem of interest must be restricted further and a potentially very difficult analysis attempted. The matrix partitioning technique of Sec. 4 may be helpful.

In the single target case, the equivalent linear model is a convenient starting point for the design of modulations which are optimized with respect to estimate variances.* A general design theory for such problems is discussed in Ref. 9 and some explicit results are given in Ref. 8. The basic idea is the use of an extension of the calculus of variations called Pontryagin's Maximum Principle to determine the optimum modulation under various constraints such as the signal's peak amplitude, total energy and bandwidth.

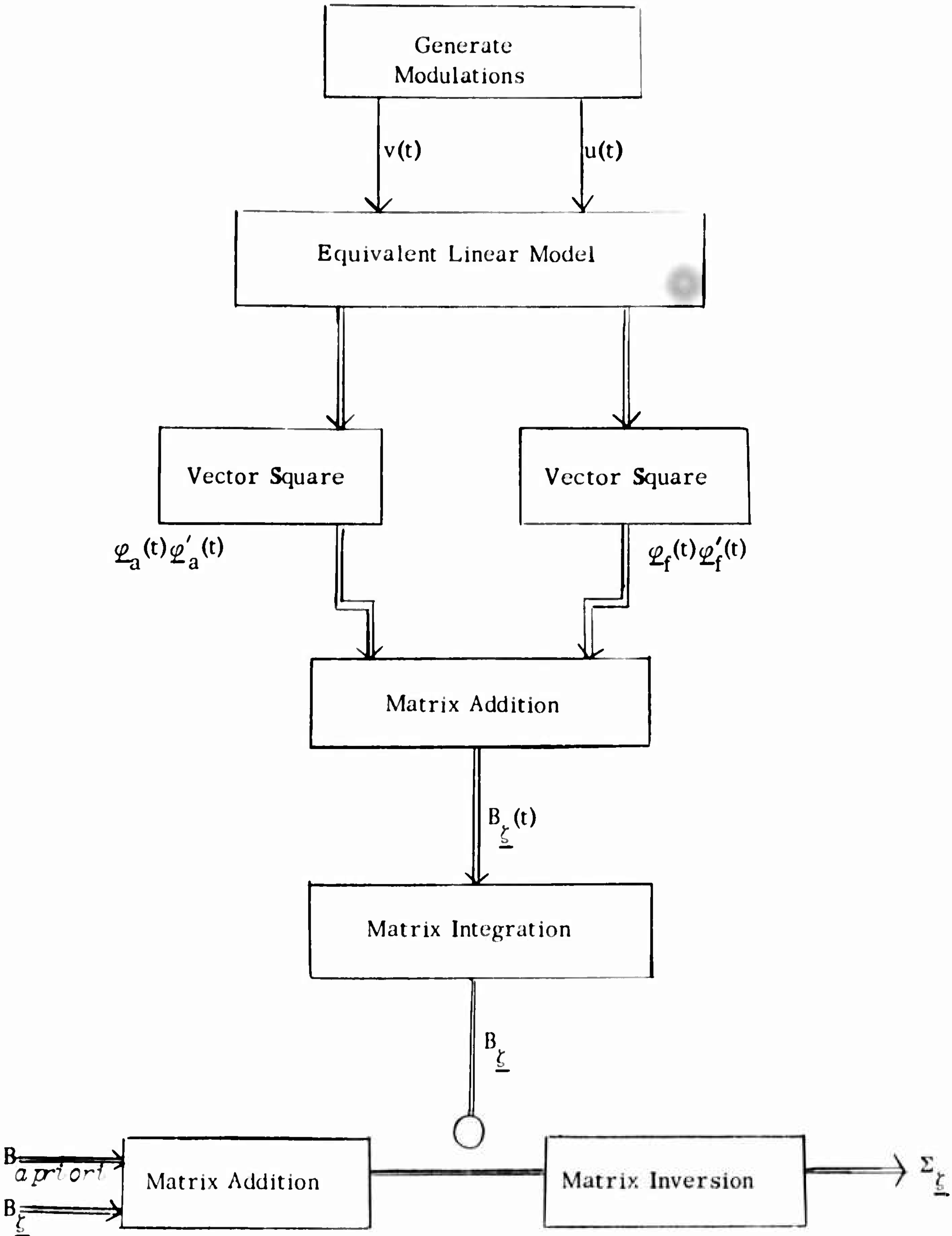
The equivalent linear model suggests a straightforward yet versatile approach to computer investigations. This is potentially of great value because of the difficulties associated with closed form results. In the single target case the computations needed to obtain the covariance matrix of the estimate errors are given by Eqs. (3.8), (3.9) and

* This is the problem which originally motivated our investigations.

(5.3) combined with the appropriate equivalent linear model, $\underline{\varphi}_a(t)$ and $\underline{\varphi}_f(t)$. One of several possible computational flows is given in Fig. 1. For the multiple target case, a possible computational flow for the equivalent model of Eqs. (6.25) and (6.26), is indicated in Fig. 2. Either figure could be programmed in a straightforward manner on a general purpose digital computer although a hybrid computer which combines integration capabilities with the general purpose arithmetic operations is even more attractive. A versatile mechanization suitable for a wide range of experimental signal designs is possible as changes in the equivalent linear model are easy and do not affect the main computational flow.

The multiple target analysis gives the estimate accuracy for a particular set of target separations. This means some range of parameter values must be investigated in order to obtain a true picture of resolution ability. The value of a convenient computational algorithm for such investigations is obvious.

Insight into the basic effects of factors such as modulation, target separation and bandwidth is a valuable quantity. Although the best basis for such insight would be simple closed form expressions for the covariance matrix, the author has found the combination of the information matrix and equivalent linear model to be very useful. It is hoped this is not an isolated phenomena.



Double lines denote vector
or matrix quantity.

Fig. 1 Computation flow for obtaining $\Sigma_{\underline{\zeta}}$, Single Target.

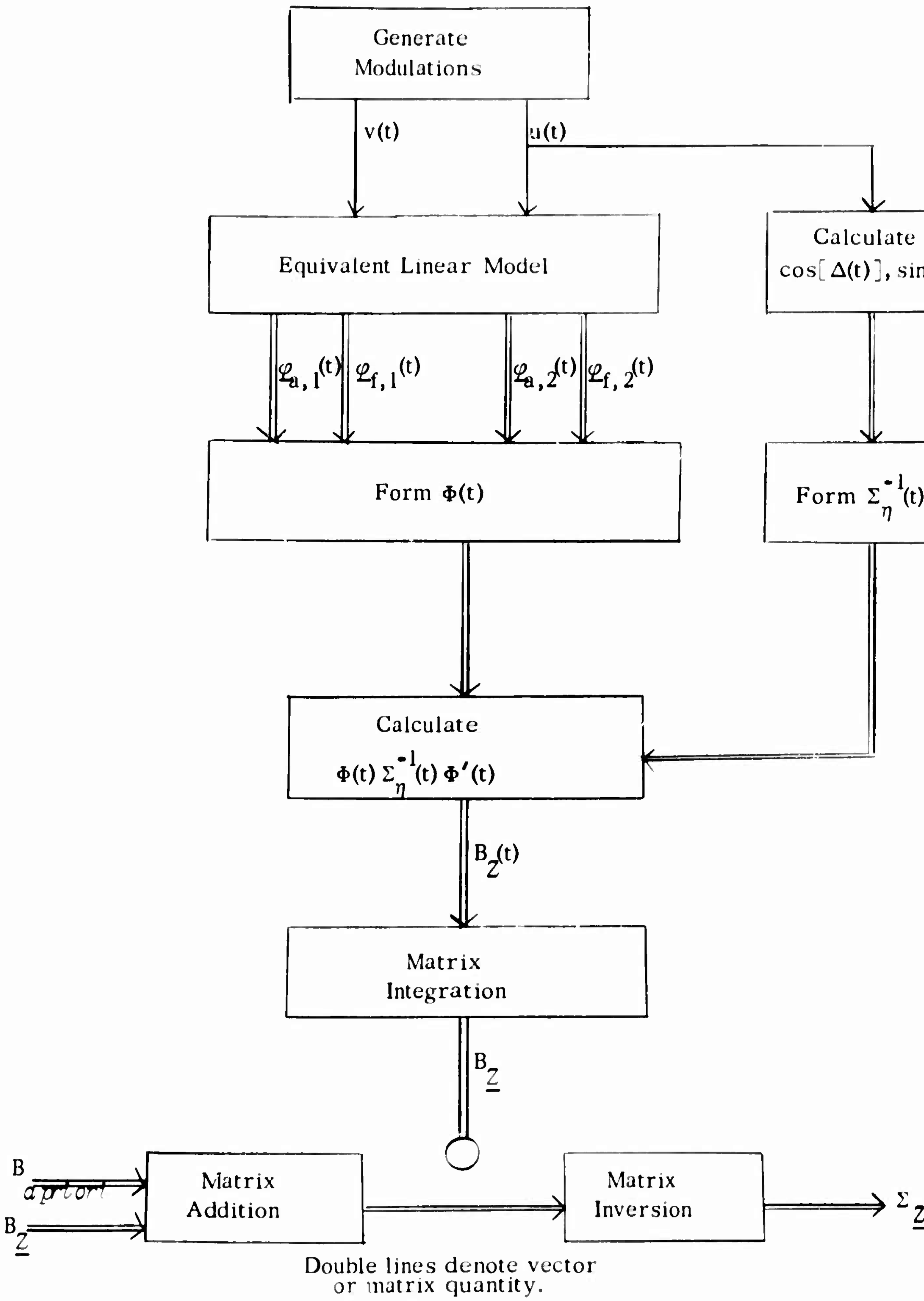


Fig. 2 Computation flow for obtaining $\Sigma_{\underline{Z}}$, Two Targets.

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